So Close, Yet So Far:
Path Integrals In Curved Space

@tsotchke

May 19, 2020

Abstract

The Path Integral Formulation of Quantum Field Theory offers as an alternative a view of particle physics based upon the historical sum of the amplitudes of time-ordered field operators. This description provides a great deal of physically descriptive power and numerically predictive for the heuristic description of the terms of the interactive Lagrangian. In this paper we will adapt this formulation for curved space, towards quantisation of particle interaction in the presence of a gravitational field. Our study will offer insight and solutions into open-ended problems in string theory and loop quantum gravity, including both supersymmetric and non-SUSY approaches. We will consider this study in terms of anomalies of the standard model and beyond as a potential foundational approach for a semi-classical description of gravitation and string theory, concluding with an optimistic view towards a pure theory of quantum gravity.

Contents

1 Introduction

Quantum Field Theory in curved space can be considered an approximation of an, as of yet, incomplete theory of quantised gravity[7]. When considering the minimal disturbance of a gravitational wave, $\widetilde{g}_{\mu\nu}$, separated off from background spacetime, $g^c_{\mu\nu}$, in Einstein's equations we have that:

$$g_{\mu\nu} = g^c_{\mu\nu} + \widetilde{g}_{\mu\nu}.$$

This approach is characterised as the "background field" method for the beginning of a foundation of quantised gravity. Taking the Einstein-Hilbert action (19) and matter field action (22) we can expand around $\widetilde{g}_{\mu\nu}$ and $g^c_{\mu\nu}$ to extract the Feynman rules for a one-loop quantum process of the lowest order. For this top level one-loop description we find that the quantisation of background gravitational field $g^c_{\mu\nu}$ takes on an equally primitive necessity as the quantisation of matter fields.

In terms of physics single closed loops correspond to infinite zero point vacuum energy which is subtracted from calculation manually via normal ordering. In flat-space this artificial operation is trivial however in curved fields the process of renormalisation is considerably more elaborate. In QED renormalisation is dependent upon the dimensionless coupling constant $e^2/\hbar c$, however gravitation depends upon G which has dimensionally quadratic length units leading to an infinite sequence of divergent terms. The expansion of these higher order terms in the powers of $\widetilde{g}_{\mu\nu}$ admit gravitons. Hence, this perturbative gravitation results in an unrenormalisable theory.

We will develop the notion of quantum field theory in curved space beginning with foundations in special relativity, constructing our machinery as we go along. Eventually we will build to the conception of the path integral formulation of QFT which we will then adapt for Einsteinian spacetime.

2 QFT in Flat Space

2.1 Klein-Gordon Equation

Classically, we may consider a real spin-0 bosonic scalar field $\phi(t, \mathbf{x})$ to be well-defined for all points $(t, \mathbf{x})$ in an n-dimensional Minkowski spacetime with Lagrangian density, $\mathcal{L}[\phi(t, \mathbf{x}), \partial_\mu \phi(t, \mathbf{x})]$, for a mass field quantisation[29]:

$$\mathcal{L} = \frac{1}{2}((\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2). \tag{1}$$

By integrating over the Lagrangian we find the action, S:

$$S = \int \mathcal{L}(x) \, \mathrm{d}^n x, \tag{2}$$

satisfying the variational principle, $\delta S = 0$, with respect to $\phi \to \phi + \delta\phi$, from which we can obtain the Klein-Gordon equation:

$$(\Box + m^2)\phi = 0. \tag{3}$$

where $\Box = g^{\mu\nu}\partial_\mu \partial_\nu$ is the D'Alembertian operator and $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric tensor[7]. For a complex spin-0 scalar field with $\phi = \phi_1 + i\phi_2$ and $\phi^* = \phi_1 - i\phi_2$ with symmetry transformations $\phi \to e^{-i\alpha}\phi$ and $\phi^* \to e^{i\alpha}\phi^*$, this Lagrangian density becomes:

$$\mathcal{L} = (\partial_\mu \phi)(\partial_\mu \phi^*) - m^2 \phi\phi^*, \tag{4}$$

where similarly the Klein-Gordon equation holds for ϕ and ϕ^* such that:

$$(\Box + m^2)\phi = 0, \quad \text{and} \quad (\Box + m^2)\phi^* = 0. \tag{5}$$

2.2 Canonical Formulation

If we consider ϕ to be a field operator, we obtain the equal-time commutation relations such that:

$$\begin{aligned}
&[\phi(t, \mathbf{x}), \phi(t, \mathbf{x}')] = 0, \\
&[\pi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = 0, \\
&[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = \delta^{n-1}(\mathbf{x} - \mathbf{x}'),
\end{aligned}$$

with canonically conjugate field momentum, π:

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi. \tag{6}$$

Expanding ϕ along the field modes, u_k, and particle annihilation and creation ladder operators, repectively a_k and $a_k^\dagger$, form a complete orthonormal basis as:

$$\phi(t, \mathbf{x}) = \sum_k [a_k u_k(t, \mathbf{x}) + a_k^\dagger u_k^*(t, \mathbf{x})]. \tag{7}$$

From this we obtain the equivalent equal-time commutation relations for the quantum states, spanning a Hilbert space in the Heisenberg picture[7]:

$$\begin{aligned}
[a_k, a_{k'}] &= 0, \\
[a_k^\dagger, a_{k'}^\dagger] &= 0, \\
[a_k, a_{k'}^\dagger] &= \delta_{kk'}.
\end{aligned}$$

2.3 Mathematics of Path Integrals

2.3.1 The Propagator

Solving boundary value problems over finite intervals is the goal of calculating expectation values of field operators in vacuum, where the free field commutator, G, is the Schwinger function and the anti-commutator, $G^{(1)}$, is Hadamard's elementary function[34]:

$$\begin{aligned}
iG(x, x') &= \langle 0|[\phi(x), \phi(x')]|0\rangle \\
&= \langle 0|\phi(x)\phi(x')|0\rangle - \langle 0|\phi(x')\phi(x)|0\rangle, \\
G^{(1)}(x, x') &= \langle 0|\{\phi(x), \phi(x')\}|0\rangle \\
&= \langle 0|\phi(x)\phi(x')|0\rangle + \langle 0|\phi(x')\phi(x)|0\rangle.
\end{aligned} \tag{8}$$

We define the Feynman propagator, G_F, as the time-ordered product of the Wightman functions, G^+ and G^-, of these free fields:

$$\begin{aligned}
iG_F(x, x') &= \langle 0|T(\phi(x)\phi(x'))|0\rangle \\
&= \theta(t - t')\,\langle 0|\phi(x)\phi(x')|0\rangle + \theta(t' - t)\,\langle 0|\phi(x')\phi(x)|0\rangle \\
&= \theta(t - t')G^+(x, x') + \theta(t' - t)G^-(x, x'),
\end{aligned} \tag{9}$$

where the Heaviside step function $\theta(t) = 1$ for all $t > 0$ and $\theta(t) = 0$ for all $t \leq 0$. For this propagator, we can use the n-dimensional solution of the wave equation in the form of a Green's function[8]. For a known initial state $u_0(\mathbf{x})$ we have that the process[37]:

$$u = u(t, \mathbf{x}) = \int_{-\infty}^{\infty} G(t, \mathbf{x}; t', \mathbf{x}') u_0(\mathbf{x}') \, \mathrm{d}\mathbf{x}'. \tag{10}$$

These functions take the form of a delta distribution, $\delta(\mathbf{x} - \mathbf{x}')$, when Fourier transformed and integrated over, for which we may define our propagator as:

$$\begin{aligned} iG(t, \mathbf{x}; t', \mathbf{x}') &= \sum_k \phi_k(\mathbf{x}) \phi_k^*(\mathbf{x}') e^{-i\mathbf{k}(t-t')} \\ \Rightarrow G_F(t, \mathbf{x}; t', \mathbf{x}') &= \frac{1}{(2\pi)^n} \int \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}') - ik^0(t-t')} \, \mathrm{d}^n\mathbf{k}}{(k^0)^2 - |\mathbf{k}|^2 - m^2}, \end{aligned} \tag{11}$$

with simple poles at $k^0 = \pm(|\mathbf{k}|^2 + m^2)^{1/2}$. In the complex scalar case this Lorentzian function can be Euclideanised along the real number line to avoid the poles in the complex plane via Wick rotation, with $\kappa = ik^0$, $\tau = -it$, and $\tau' = -it'$, to obtain the relationship[7]:

$$\begin{aligned} G_F(t, \mathbf{x}; t', \mathbf{x}') &= -iG_E(i\tau, \mathbf{x}; i\tau', \mathbf{x}') \\ &= -\frac{i}{(2\pi)^n} \int \frac{(e^{i\kappa(\tau-\tau')} \, \mathrm{d}\kappa)(e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \, \mathrm{d}^{n-1}\mathbf{k})}{\kappa^2 + |\mathbf{k}|^2 + m^2}, \end{aligned} \tag{12}$$

satisfying the Klein-Gordon equation such that the product with the now Euclidean Green Function gives:

$$(\Box_{\mathrm{E}_x} - m^2) G_E(x, x') = -\delta^{n-1}(x - x'), \tag{13}$$

where the D'Alembertian acting on n-dimensional Euclidean space is the elliptic operator, $\Box_{\mathrm{E}} = \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial (x_1)^2} + ... + \frac{\partial^2}{\partial (x_{n-1})^2}$.

2.3.2 Path Integral Quantisation

By our canonical formalism, we have the commutative relationship:

$$[\phi(x), \phi(x')] = iG(x, x'), \tag{14}$$

which also holds for the particle creation and annihilation operators, owing to the equivalency between covariant and canonical formalisms in globally hyperbolic spacetime[7]. However, if we wish to consider the fundamental action of a quantised field it necessitates an alternative approach. For this, we consider the fundamental object of the theory of path integrals, the functional Z, which for a conserved current, J_μ, of a measured scalar field, ϕ, in vacuum takes the form:

$$Z[J] = {}_{out}\langle 0|0\rangle_{in} = \int \mathcal{D}[\phi] e^{iS[\phi]+i\int \mathrm{d}^n x J(x)\phi(x)} \tag{15}$$

such that $Z[0] = \langle 0|0\rangle = \mathbb{1}$ is normalised to unity[27]. If we take the discretised variant of our Lagrangian action (2) as a function of the free field ϕ in our Lagrangian density (1), while integrating by parts, the action becomes:

$$S[\phi] = \int \mathrm{d}^n x \left[-\frac{1}{2}\phi(\Box + m^2 - i\epsilon)\phi \right], \tag{16}$$

which redefines our functional as:

$$\begin{aligned} Z[J] &= -\frac{1}{2}\int \mathrm{d}^n x\, \mathrm{d}^n y\, \phi(x)(\Box_x + m^2 - i\epsilon)\delta^n(x-y)\phi(y) + \int J(x)\phi(x)\,\mathrm{d}^n x \\ &= -\frac{1}{2}\int \mathrm{d}^n x\, \mathrm{d}^n y\, \phi(x) K_{xy}\phi(y) + \int J(x)\phi(x)\,\mathrm{d}^n x, \end{aligned} \tag{17}$$

where the symmetric operator, $K_{xy} = (\Box_x + m^2 - i\epsilon) = -G_F^{-1}(x,y)$, is the Feynman kernel as the inverse of the propagator, well-defined across all paths of $\mathcal{D}[\phi]$, the functional integral[3]. This functional can be further expressed as the numerical quantity of a Gaussian type, proportional to product of the fractional Jacobian of the kernel by the action of the current:

$$Z[J] \propto e^{\frac{1}{2}\mathrm{Tr}(-G_F)} e^{-\frac{1}{2}\int \mathrm{d}^n x\, \mathrm{d}^n y J(x) G_F(x,y) J(y)}. \tag{18}$$

3 QFT in Curved Space

3.1 Einsteinian Spacetime Action

General Relativity has been mathematically formalised by many authors in the years since its foundation in the early 20th century[25]. The foundation of our interest in curved space path integrals rests upon the Wick-rotated Euclidean Einstein-Hilbert action in vacuum[19]:

$$S_{\mathrm{E}}[g] = -\frac{c^4}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{g}(R-2\Lambda) - \frac{c^4}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^3x\,\sqrt{h}K. \tag{19}$$

The first integral covers a region $\mathcal{M}$ on a 4-dimensional spacetime manifold with $g = \det g^{\mu\nu}$ for basis tensors $\tilde{e}^{\,\mu}\otimes\tilde{e}^{\,\nu}$ such that $g^{\mu\nu} = \vec{e}^{\,\mu}\cdot\vec{e}^{\,\nu}$. The second corresponds to its space-like boundary $\partial\mathcal{M}$ with $h = \det h_{\mu\nu}$ which is the 3-dimensional metric corresponding to the velocity on the boundary such that[24]:

$$h_{\mu\nu} = g_{\mu\nu} + n^{\mu}n_{\mu}, \tag{20}$$

where $g_{\mu\nu} \neq g^{\mu\nu}$ (generally speaking) and n_μ is the unit normal to $\sum_t$ with $n_\mu n^\mu = -1$. K is the 'second fundamental form' or 'extrinsic curvature' of the purely spatial symmetric tensor associated with the 'velocity' of $h_{\mu\nu}$:

$$K_{\mu\nu} = h_{\mu}^{\ \rho}\nabla_\rho n_\nu = \frac{1}{2}\mathcal{L}_{\mathbf{n}}h_{\mu\nu}, \tag{21}$$

where $\mathbf{n}$ denotes the unit normal vector field and $K_{\mu\nu}n^\mu = K_{\nu\mu}n^\nu = 0$, with $\mathrm{Tr}\,K_{\mu\nu} = K := K_{\mu}^{\ \mu} = h^{\mu\nu}K_{\mu\nu} = \theta$. To find the complete matter equation we must also consider the action of non-vacuum matter fields given by the 'matter field action'[10]:

$$S_{\mathrm{M}} = -\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{g}\left[g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)\right], \tag{22}$$

from which we derive the equation of motion:

$$\Box\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \tag{23}$$

We vary the matter action with respect to the variation in the metric to obtain the 'source' of the gravitational field, the energy-momentum tensor:

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}}, \tag{24}$$

from which the variation of $S_{\mathrm{E}} + S_{\mathrm{M}}$ recovers the complete Einstein field equation[19]:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{25}$$

which describes the reaction of the curvature of spacetime to the presence of matter's energy-momentum where[10]:

$$\begin{aligned} T^{(\phi)}_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} \\ &= (\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}(\partial_\rho\phi)(\partial_\sigma\phi) - g_{\mu\nu}V(\phi). \end{aligned} \tag{26}$$

3.2 Conformal Field Theory

For a coordinate-invariant field theory in curved space we must introduce a curvature invariant Lagrangian density with respect to a scalar field, ϕ[10]:

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \xi R\phi^2\right), \tag{27}$$

where the conformal coupling, $\xi(n) \equiv \frac{(n-2)}{4(n-1)}$, is dimensionless, taking on a minimal interactive coupling with $\xi = 0$, and $\xi = \frac{1}{6}$ in 4-dimensions. We may now define our curved space canonical momentum operator (6) as:

$$\pi = \sqrt{-g}\partial_t\phi, \tag{28}$$

with our canonical commutation relations becoming:

$$\begin{aligned} &[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] = 0, \\ &[\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = 0, \\ &[\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = \frac{i}{\sqrt{-g}}\delta^{n-1}(\mathbf{x} - \mathbf{x}'). \end{aligned}$$

For a non-vanishing spacetime dependent function, $\omega(x)$, we define a conformal transformation as a local change of scale measured by the metric element:

$$\widetilde{ds}^2 = \omega^2(x)ds^2. \tag{29}$$

We make special note of the fact that null curves are left-invariant by conformal transformations, such that in our metric:

$$\widetilde{g}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \omega^2(x)g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{30}$$

where $\widetilde{g}_{\mu\nu} = \omega^2(x)g_{\mu\nu}$, from which we obtain the D'Alembertian of ϕ as[7]:

$$\begin{aligned}\Box\phi &= \omega^2\widetilde{\Box}\phi - (n-2)\widetilde{g}^{\alpha\beta}\omega(\partial_\alpha\omega)(\partial_\beta\phi),\\ \widetilde{\Box}\phi &= \omega^{-2}\Box\phi + (n-2)g^{\alpha\beta}\omega^{-3}(\partial_\alpha\omega)(\partial_\beta\phi),\end{aligned} \tag{31}$$

leading to the scalar field equation of motion:

$$\left[\Box_x + m^2 + \xi R(x)\right]\phi(x) = 0, \tag{32}$$

with complex solutions on the inner product for a space-like hypermanifold Σ on $\gamma_{ij} = -g_\Sigma(x)$, the induced metric[32]:

$$(\phi_1, \phi_2) = -i\int_\Sigma(\phi_1\partial_\mu\phi_2^* - \phi_2\partial_\mu\phi_1^*)n^\mu\sqrt{\gamma}\,\mathrm{d}\Sigma^\mu, \tag{33}$$

for the unit normal n^μ. For orthonormal conjugate modes $f_i(x^\nu)$ and $g_i(x^\nu)$, we may expand the fields in terms of annihilation and creation operators as:

$$\phi = \sum_i\left(\hat{a}_i f_i + \hat{a}_i^\dagger f_i^*\right) = \sum_i\left(\hat{b}_i g_i + \hat{b}_i^\dagger g_i^*\right), \tag{34}$$

from which we derive the so-called Bogolubov transformation matrices[11]:

$$g_i = \sum_j\left(\alpha_{ij}f_j + \beta_{ij}f_j^*\right), \quad \text{and} \quad f_i = \sum_j\left(\alpha_{ji}^* g_j + \beta_{ji}g_j^*\right), \tag{35}$$

where $\alpha_{ij} = (g_i, f_j)$ and $\beta_{ij} = -(g_i, f_j^*)$. We may then define these particle intraoperational mode transformations as:

$$\hat{a}_i = \sum_j\left(\alpha_{ji}\hat{b}_j + \beta_{ji}^*\hat{b}_j^\dagger\right), \quad \text{and} \quad \hat{b}_i = \sum_j\left(\alpha_{ij}^*\hat{a}_j - \beta_{ij}^*\hat{a}_j^\dagger\right). \tag{36}$$

Finally, the geodesic distance will vanish at null-separated points on a Lorentzian manifold such that:

$$\sigma_\epsilon(x, x') = \lim_{\epsilon\to 0}\left(\sigma(x, x') + 2i\epsilon(t - t') + \epsilon^2\right), \tag{37}$$

with a two-point function, $G(x, x') = \langle\psi|\phi(x)\phi(x')|\psi\rangle$, for a field, ϕ, and quantum state, ψ, defined as:

$$G^{(1)}(x, x') = \frac{U(x, x')}{4\pi^2\sigma_\epsilon} + V(x, x')\ln\sigma_\epsilon + W(x, x'), \tag{38}$$

for regularised Hadamard state functions $U, V,$ and W.

3.3 Path Integrals in Curved Space

Beginning with the Euclidean time classical action for fields x^μ with arbitrary transition amplitudes, A_μ we have that[4]:

$$S = \int_{t_i}^{t_f} \mathrm{d}t \left[\frac{1}{2} g_{\mu\nu}(x)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + iA_\mu(x)\frac{dx^\mu}{dt} + V(x)\right]. \tag{39}$$

Rescaling the classical action in terms of ghost fields, a^μ, b^μ, and c^μ, and counter-term, V_{DR}, for a time parameter $-1 \leq \tau \leq 0$ we may obtain the complete action in $(D+1)$-dimensions as[6]:

$$S = \int_{-1}^{0} \mathrm{d}^{D+1}\tau \left\{\frac{1}{2} g_{\mu\nu}(x)\left(\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + a^\mu a^\nu + b^\mu c^\nu\right) + i\beta A_\mu(x)\frac{dx^\mu}{d\tau} + \beta^2\left(V(x) + V_{\mathrm{DR}}(x)\right)\right\}, \tag{40}$$

for the total propagation time, β. Expanding all of the quantum ghost fields, including an additional quantised field, q^μ, we can define the total scalar field as $\phi^\mu(\tau) = \sum_{\rho=1}^{\infty} \phi^\mu_\rho \,\mathrm{Sin}\,\pi\rho\tau$, where the functional across all paths is defined as the coefficient space of Fourier transformations such that we have the path integral:

$$\left\langle x_f^\lambda, t_f | x_i^\lambda, t_i\right\rangle = \int_{\mathrm{BC}} \mathcal{D}q\ \mathcal{D}a\ \mathcal{D}b\ \mathcal{D}c\ e^{-\frac{1}{\beta}S} \tag{41}$$

$$x^\mu(\tau) = x^\mu_{\mathrm{bg}}(\tau) + q^\mu(\tau) \tag{42}$$

$$\Rightarrow \mathcal{D}q\ \mathcal{D}a\ \mathcal{D}b\ \mathcal{D}c = A \prod_{\lambda=1}^{\infty}\prod_{\rho=1}^{\mu} \lambda\, \mathrm{d}q^\rho_\lambda\, \mathrm{d}a^\rho_\lambda\, \mathrm{d}b^\rho_\lambda\, \mathrm{d}c^\rho_\lambda, \tag{43}$$

where x^μ_{bg} is a background trajectory, and A is an arbitrary amplitude, often defined as the Feynman measure with $A = (2\pi\beta)^{-n/2}$.

3.4 Semi-Classical Theory of Gravity

In the early development of quantum mechanics it was often necessary to consider the electromagnetic field as a classical background embedding for the quantisation of matter. Analogously, this logic may be applied in pursuit of a theory of quantum gravity: in which a semi-classical description is given by quantising matter fields over the backdrop of a classical gravitational field[7].

This "half-classical" and "half-quantised" point-of-view is most often represented as a continuous gravitational field coupled to a discretised matter field as[35]:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \langle T_{\mu\nu} \rangle \,, \tag{44}$$

however this theory suffers from serious drawbacks and limitations, some of which are mitigated if the fluctuations in the energy-momentum stress tensor are very small such in the case that:

$$\langle T_{\alpha\beta}(x) T_{\mu\nu}(y) \rangle \approx \langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle \,. \tag{45}$$

If we consider a linearised perturbation of curved (but nearly flat) space-time as a metric fluctuation, we obtain the retarded Euclidean time Green's function as the expectation value of the quantum eigenstate:

$$\langle G_{\text{ret}}(x, x') \rangle = \frac{\theta(t - t')}{8\pi^2} \int_{-\infty}^{\infty} \mathrm{d}\alpha \, e^{i\alpha\sigma_0 - \frac{1}{2}\alpha^2 \langle \sigma_1^2 \rangle}, \tag{46}$$

where $\sigma = \sigma_0 + \sigma_1 + O(h_{\mu\nu}^2)$ is the squared separation in the presence of the perturbation. This integral converges only in the case that $\langle \sigma_1^2 \rangle > 0$, in which case the Hadamard function is defined as[15]:

$$\left\langle G^{(1)}(x, x') \right\rangle = -\frac{1}{2\pi^2} \left\langle \frac{1}{\sigma} \right\rangle = -\frac{1}{2\pi^2} \int_0^{\infty} \mathrm{d}\alpha \; \mathrm{Sin} \, \alpha\sigma_0 \; e^{-\frac{1}{2}\alpha^2 \langle \sigma_1^2 \rangle}, \tag{47}$$

where in the time-like limit, $\sigma_0^2 \gg \langle \sigma_1^2 \rangle$, we have $\left\langle G^{(1)}(x, x') \right\rangle \sim -\frac{1}{2\pi^2} \frac{1}{\sigma_0}$, and in the light-like limit, $\sigma_0^2 \ll \langle \sigma_1^2 \rangle$, we have that $\left\langle G^{(1)}(x, x') \right\rangle \sim -\frac{\sigma_0}{2\pi^2 \langle \sigma_1 \rangle}$ is finite. Likewise, the Feynman propagator is also finite in the light-like limit such that[11]:

$$G_F(x, x') = \frac{1}{2} [G_{\text{ret}}(x, x') + G_{\text{ret}}(x', x)] - iG^{(1)}(x, x'). \tag{48}$$

4 Gravitational Anomalies

Anomalies are quantum effects which break gauge symmetries associated with the motion equations of classical physics[21]. The final section of our paper is concerned with anomalies associated with gravitational effects in a variety of contexts, with an emphasis on anomalies due to loops with tensor fields coupled to external gravity[6].

4.1 Spin 2 Particles

A graviton field can be described as an irreducible representation of linearised perturbations upon a classical spacetime background[7]. For a massive spin-2 field with five polarisations comprised of two transverse and three longitudinal components transform the canonical momentum as $\pi_\mu = \pi_\mu^T + \pi_\mu^L$ with $\partial_\mu \pi_\mu^T = 0$. From this, we obtain the Lagrangian density[29]:

$$\mathcal{L} = \frac{1}{2} h_{\mu\nu} \Box h_{\mu\nu} - h_{\mu\nu} \partial_\mu \partial_\alpha h_{\nu\alpha} + h \partial_\mu \partial_\nu h_{\mu\nu} - \frac{1}{2} h \Box h + \frac{1}{2} m^2 (h_{\mu\nu}^2 - h^2), \quad (49)$$

for some symmetric tensor $h_{\mu\nu} = h_{\mu\nu}^T + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu$, which we may parametrise in terms of ξ^α as:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + (\partial_\mu \xi^\alpha) h_{\alpha\nu} + (\partial_\nu \xi^\alpha) h_{\mu\alpha} + \xi^\alpha \partial_\alpha h_{\mu\nu}. \quad (50)$$

4.2 Examples of Anomalies

4.2.1 Chiral Anomaly For Gravity in $(4k + \epsilon)$-Dimensions

For a massless Dirac fermion, λ, we may define the Lagrangian in n Euclidean dimensions as[15]:

$$\mathcal{L} = -e \overline{\lambda} e_\nu^\mu \gamma^\nu D_\mu \lambda, \quad (51)$$

with $\overline{\lambda} = \lambda^\dagger i \gamma^0$, $e = \det e_\mu^\nu$, and $D_\mu \lambda = \partial_\mu \lambda + \frac{1}{4} \omega_{\mu\nu\rho}(e) \gamma^\nu \gamma^\rho$. If one defines the path integral as:

$$Z[e_\mu^\nu] = \int \mathcal{D}\lambda \mathcal{D}\overline{\lambda} e^{-\int \mathrm{d}^n x \mathcal{L}}, \quad (52)$$

then the chiral anomaly cases a breakdown of the gauge symmetry for the effective action under $\delta A_\mu = \partial_\mu \alpha$ such that an anomaly is generated by the regulated Jacobian, Tr J, given by infinitesimal chiral transformation $J = i\alpha\gamma^5$[6]:

$$\mathrm{An} = \lim_{\beta\to 0} \mathrm{Tr}\, J e^{-\beta\mathcal{R}}, \tag{53}$$

where $\mathcal{R} = -\frac{1}{2} g^{\frac{1}{4}} \not{D}^2 g^{-\frac{1}{4}} = -\frac{1}{2} g^{\frac{1}{4}} D_\mu \sqrt{g} g^{\mu\nu} D_\nu g^{-\frac{1}{4}} - \frac{1}{8} R$. Under covariant transformation in response to the effective action Γ we find an anomaly:

$$\mathrm{An}_{\mathrm{cov}}(\xi) = \delta_{\mathrm{cov}}(\xi)\Gamma[e_\mu{}^\nu] = -\int \mathrm{d}x\, e\, \xi^\nu (D_\mu T_\nu{}^\mu), \tag{54}$$

where $T_\nu{}^\mu = \frac{1}{e}\frac{\delta\Gamma}{\delta e_\mu{}^\nu(x)}$ is the effective energy-momentum tensor. This stress tensor diverges under covariant transformation.

4.2.2 Cancellation of Anomalies in IIB Supergravity

The gravitational anomalies for complex chiral spin-$\frac{1}{2}$, chiral spin-$\frac{3}{2}$, and an anti-symmetric tensor fields are found to be[6]:

$$\mathrm{An}_{(\mathrm{g,\ spin}\text{-}\frac{1}{2})} = \int e^{\frac{1}{2}} \ln \left[\frac{\tilde{R}/4}{\mathrm{Sinh}\, \tilde{R}/4}\right] \tag{55}$$

$$\mathrm{An}_{(\mathrm{g,\ spin}\text{-}\frac{3}{2})} = \int \left[\mathrm{Tr}\, e^{\tilde{R}/2} - 1\right] e^{\frac{1}{2}} \ln \left[\frac{\tilde{R}/4}{\mathrm{Sinh}\, \tilde{R}/4}\right] \tag{56}$$

$$\mathrm{An}_{(\mathrm{g,\ AT})} = -\frac{1}{8}\int e^{\frac{1}{2}} \ln \left[\frac{\tilde{R}/4}{\mathrm{Tanh}\, \tilde{R}/4}\right]. \tag{57}$$

If we simplify the notation by the introduction of y^n in the expansion of the trigonometric series, we have:

$$\mathrm{An}_{\frac{1}{2}} = e^{-\frac{1}{2}\mathrm{Tr}\ \ln\left(1+\frac{1}{3!}y^2+\frac{1}{5!}y^4+...\right)} \tag{58}$$

$$\mathrm{An}_{\frac{3}{2}} = \left[\mathrm{Tr}\ \left(1 + 2y^2 + \frac{2}{3}y^4 + \frac{4}{45}y^6 + ...\right)\right] \mathrm{An}_{\frac{1}{2}} \tag{59}$$

$$\mathrm{An}_{\mathrm{AT}} = -\frac{1}{8} e^{-\frac{1}{2}\mathrm{Tr}\ \ln\left(1-\frac{4}{3}y^2+\frac{32}{15}y^4-...\right)}, \tag{60}$$

such that $\mathrm{An}_{\frac{3}{2}} - \mathrm{An}_{\frac{1}{2}} + \mathrm{An}_{\mathrm{AT}} = 0$ for all sixth order terms in $\tilde{R}$ with $n = 10$.

5 Conclusion

We have seen that the application of path integrals to perturbative gravitational problems yields very interesting and utile results. However, due to high order terms we find many of these approaches to be divergent, producing non-renormalisable theories. While some anomalies are correctable through the application of supergravity, there remains a great deal of interest in studying non-perturbative gravitational theories based upon the notion of discretised lattice-oriented spacetime geometries[22].

For a Euclidean path integral with Einstein-Hilbert action (19) in a lattice gauge we consider the functional[19]:

$$Z[g] = \int \mathcal{D}g_{\mu\nu}\, e^{iS_E[g_{\mu\nu}]}, \tag{61}$$

where the action is unbounded under metric conformal transformation:

$$S_{\mathrm{E}}[\tilde{g}] = -\frac{c^4}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{g}(\omega^2 R + 6\omega_\mu\omega_\nu g^{\mu\nu} - 2\Lambda\omega^4) - \frac{c^4}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^3x\,\sqrt{h}\omega^2 K, \tag{62}$$

leading to a divergence in large scale gradients of ω^2. However, this so-called conformal factor problem has been argued to be cancellable by the divergence of a similar, but opposite signed, term in the measure of the path integral. For this reason, we may redefine our functional as a summation of discrete spacetime geometries:

$$Z \to \sum_{T\in\mathcal{T}} \frac{1}{C_T} e^{iS(T)}, \tag{63}$$

where $S(T)$ is the discretised Regge action over sub-geometries T. This leads to a view of spacetime as a history of piecewise discrete topologies[23]. This approach leads to a computational model of quantised gravity that is both background independent and completely renormalisable[1].

Regardless, the path integral approach feels intuitively and theoretically viable for the production of quantised gravitational models across a variety of physical worldviews. We hope that you have developed somewhat of a taste for the functional approach to Lagrangian dynamics in curved spacetime. Perhaps with some additional effort it will be possible to produce an ultimate union between the non-renormalisable perturbative and renormalisable non-perturbative approaches to quantum gravity in a manner befitting both their theoretical elegance as distinct interpretations, and potential predictive capacity as prototypical solutions to one of the oldest problems in the history of modern physics: we are yet so close, but so far.

References

[1] J. Ambjorn, J. Gizbert-Studnicki, A. Goerlich, J. Jurkiewicz, and R. Loll, *Renormalization in quantum theories of geometry*, 2020.

[2] Alessio Baldazzi, Roberto Percacci, and Vedran Skrinjar, *Quantum Fields without Wick Rotation*, Symmetry **11** (2019), no. 3, 373.

[3] J. Barcelos-Neto and Ashok Das, *Path integrals and the solution of the Schwinger model in curved space-time*, Physical Review D **33** (1986), no. 8, 2262–2266.

[4] Fiorenzo Bastianelli and Olindo Corradini, *Mode regularization of the configuration space path integral in curved space*, Physical Review D **60** (1999), no. 4, 044014.

[5] Fiorenzo Bastianelli, Olindo Corradini, and Laura Iacconi, *Simplified path integral for supersymmetric quantum mechanics and type-a trace anomalies*, Journal of High Energy Physics **2018** (2018), no. 5, 010.

[6] Fiorenzo Bastianelli and Peter van Nieuwenhuizen, *Path Integrals and Anomalies in Curved Space*, Cambridge University Press, 2009.

[7] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, 1999.

[8] Frederick W. Byron and Robert W. Fuller, *Mathematics of Classical and Quantum Physics*, Dover Publication, 2012.

[9] P. Candelas and D. J. Raine, *Feynman propagator in curved space-time*, Physical Review D **15** (1977), no. 6, 1494–1500.

[10] Sean M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Cambridge University Press, 2019.

[11] Bryce S. DeWitt, *Quantum Field Theory in Curved Spacetime*, North-Holland Publishing Company, 1975.

[12] J. C. Dolivo and M. Torres, *The canonical formalism and path integrals in curved spaces*, Journal of Physics A: Mathematical and General **21** (1988), no. 16, 3355–3363.

[13] Anthony Duncan, *The Conceptual Framework of Quantum Field Theory*, Oxford University Press, 2018.

[14] Rafael Ferraro and Mauricio Leston, *Phase Space Path Integral In Curved Space*, International Journal of Modern Physics A **16** (2001), no. 31, 5033–5042.

[15] Kazuo Fujikawa and Hiroshi Suzuki, *Path Integrals and Quantum Anomalies*, Clarendon Press, 2013.

[16] Herbert W. Hamber, *Quantum Gravitation: The Feynman Path Integral Approach*, Springer, 2010.

[17] Gerard 't Hooft, *Perturbative Quantum Gravity*, From Quarks and Gluons to Quantum Gravity: Proceedings of the International School of Subnuclear Physics, Erice, Sicily, Italy, August - September 2002 (Antonino Zichichi, ed.), The Subnuclear Series, vol. 40, World Scientific, 2003, pp. 249–269.

[18] Steven Johnston, *Particle propagators on discrete spacetime*, Classical and Quantum Gravity **25** (2008), no. 20, 202001.

[19] Claus Kiefer, *Quantum Gravity*, Oxford University Press, 2012.

[20] H. Kleinert, *Quantum Equivalence Principle*, Functional Integration NATO ASI Series (1997), 67–92.

[21] Paul Langacker, *The Standard Model and Beyond*, CRC Press, 2017.

[22] Adrian P. C. Lim, *Quantized Curvature in Loop Quantum Gravity*, Reports on Mathematical Physics **82** (2018), no. 3, 355–372.

[23] Renate Loll, *Quantum Gravity from Causal Dynamical Triangulations: A Review*, Classical and Quantum Gravity **37** (2019), no. 1, 013002.

[24] David Lovelock and Hanno Rund, *Tensors, Differential Forms and Variational Principles*, Dover Publications, 1989.

[25] Charles W. Misner, *Feynman Quantization of General Relativity*, Reviews of Modern Physics **29** (1957), no. 3, 497–509.

[26] Leonard Emanuel Parker and David J. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*, Cambridge University Press, 2009.

[27] Walter Rudin, *Functional Analysis*, McGraw-Hill, 2006.

[28] Reiner Kurt Sachs and Hung-Hsi Wu, *General Relativity for Mathematicians*, Springer, 1977.

[29] Matthew D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press, 2018.

[30] Mark R. Sepanski, *Compact Lie Groups*, Springer, 2011.

[31] Singh, Dinesh and Mobed, Nader, *A New Perspective on Path Integral Quantum Mechanics in Curved Space-Time*, EPJ Web of Conferences **58** (2013), 01011.

[32] Michael Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. 1 & 2, Publish or Perish, Inc, 2005.

[33] Mark Srednicki, *Quantum Field Theory*, Cambridge University Press, 2018.

[34] David J. Toms, *The Schwinger Action Principle and Effective Action*, Cambridge University Press, 2012.

[35] Robert M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, University of Chicago Press, 2008.

[36] Steven Weinberg, *The Quantum Theory of Fields: Supersymmetry*, vol. 3, Cambridge University Press, 2009.

[37] Eberhard Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Springer-Verlag, 1995.