Fluid Hang the Twoddle of the Fuddled: Quantum Spin Liquids and Majorana Fermions in Topological Superconductors

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Abstract

Topological Insulators have inherent properties that make them extremely useful in new material development for electronic devices. Current research has demonstrated experimentally the existence of adiabatically non-degenerate superconduction modes at the interface of Topological Insulators and conventional Superconductors. These Topological Superconduction modes can be shown to be quasiparticle excitations obeying Majorana Fermion statistics; such that these anti-particle modes are equivalent to their particle modes at the boundary of material interface. Within materials, these Fermionic excitations behave as a Quantum Spin Liquid. We will examine Topological Insulators and Superconductors, Majorana Fermions and their role in superconduction; which may have applications within topological quantum computing.

1 Introduction

Since even before the discovery of the Quantum Hall Effect it had been known that the dichotomy of metals and insulators is inadequate to explain the many observed phenomena of solid-state physics. By the Landauer-Bütticker formalism a two-dimensional metal is known in the presence of such a strong magnetic field to induce the motions of electrons into confinement within the bulk. However, conduction along the surface or the edges is permitted by means of a number of gapless, but discrete, completely open delocalised state channels. It has been demonstrated through artificial systems research that the non-trivial topology of the occupied bands could be shown without the presence of a strong magnetic field.

2 Non-Commutative Geometry

In the pursuit of a consistent mathematical framework for the phenomenology of Topological Insulators, we are motivated to consider the Integer Quantum Hall Effect[2]. It is in the consideration of this example that the groundwork for the description of our non-commutative model will be based. In a tightly bound discrete Quantum Hall System the model consists of a boundary free Hamiltonian $\mathcal{H} = \ell(\mathbb{Z}^2)$ with $\hat{U}$ and $\hat{V}$ magnetic translations acting with unitary operators U and V such that $H = U + U^* + V + V^*$. In the Landau Gauge we have that:

$$\begin{aligned} (U\lambda)(m,n) &= \lambda(m-1,n), \qquad (V\lambda)(m,n) = e^{-2\pi i\phi m}\lambda(m,n-1), \\ (\hat{U}\lambda)(m,n) &= e^{-2\pi i\phi n}\lambda(m-1,n), \qquad (\hat{V}\lambda)(m,n) = \lambda(m,n-1) \end{aligned} \tag{1}$$

where ϕ is the magnetic flux through the unit cell $\lambda \in \ell(\mathbb{Z}^2)$ such that $\hat{U}\hat{V} = e^{-2\pi i\phi}\hat{V}\hat{U}$, and the rotation algebras are defined as $A_\phi \cong C^*(U,V)$ and $A_{-\phi} \cong C^*(\hat{U},\hat{V})$. Considering the inverse algebra in terms of matrix multiplication $(ab)^{-1} = b^{-1}a^{-1}$ such that $A_{-\phi} \cong A_\phi^{-1}$ then we can define the non-commutative operation on two unitary matrices exactly as:

$$U^{-1}V^{-1} = (VU)^{-1} = e^{-2\pi i\phi}(UV)^{-1} = e^{-2\pi i\phi}V^{-1}U^{-1} \tag{2}$$

where the choice of gauge implements an automorphism $\eta(U^m) = V^*U^mV$ on $C^*(U,V) \cong C^*(U) \rtimes_\eta \mathbb{Z}$. From here we obtain the topological properties of the Quantum Hall Effect, namely the Hamiltonian and the non-commutative geometry of the Brillouin zone.

3 Topological Insulators and Superconductors

The existence of the invariant topological Chern number connected to the number of edge state channels of the occupied bands dons the notion of Topological Insulators. We can define the Chern Invariant[3] in terms of the line integral $\mathcal{A}_m = i\langle u_m|\nabla_k|u_m\rangle$ of Bloch wave functions $|u_m(\mathbf{k})\rangle$, and the surface integral $\mathcal{F} = \nabla \times \mathcal{A}_m$ as the total flux in the Brillouin Zone:

$$\nu_m = \frac{1}{2\pi}\int \mathcal{F}_m \mathrm{d}^2\mathbf{k}, \tag{3}$$

such that $\nu = 0$ for all Time-Reversal invariant Bloch Hamiltonians.

Topological Insulators have bulk energy band gaps in the highest occupied bands like conventional insulators but permit protected conduction states consisting of odd numbers of Dirac Fermions along gapless channels along the edges or the surface itself[6] due to a mixture of spin orbit interactions and Time-Reversal symmetry[3]. A classification of Topological Insulators and Superconductors in terms of the their underlying symmetries may be Quantum Hall State (no symmetry), $\mathbb{Z}_2$ Topological Insulators, and $\mathbb{Z}_2$ and $\mathbb{Z}$ Topological Superconductors.

3.1 Topological Insulators

We can consider Topological Insulators in terms of their symmetries which evoke topologically invariant properties[2]. Conventionally, these are the Time-Reversal, Particle-Hole (commonly, Charge-Conjugation), and Chiral Sublattice symmetries. From the topological example of the IQHE we may also abstractly express the Hall conductance as a homology class of pairings over the momentum space of a crystal sample. Since this effect is linked to intrinsic topological information the IQHE can be considered as a simple Topological Insulator without any of these other symmetries.

The so-called Quantum Spin-Hall Effect had been predicted originally in graphene research but was not verified until somewhat recent experimental precision allowed for the accurate measurement to confirm its existence. This effect consists of the spin-oriented edge channels giving two opposing currents of states spin-up and -down such that, though the net current is zero, there is a non-trivial conductance for each spin-component linked to the topological invariants of classical fibre bundles over the Brillouin zone. Thus we can consider the Quantum Spin Hall Insulator as a Topological Insulator.

Hall conductivity is odd under $\mathcal{T}$ which gives non-trivial and distinct topological insulating band states when $\mathcal{T}$ is broken. We may define a $\mathcal{T}$ invariant Bloch Hamiltonian which satisfies $\Theta\mathcal{H}(\mathbf{k})\Theta^{-1} = \mathcal{H}(-\mathbf{k})$ in terms of equivalence classes of this constraint[3].

3.2 Superconductivity

Band theory permits us to classify superconductors topologically. Bardeen-Cooper-Schrieffer mean field theory describes a Hamiltonian for such a spin-free system of electrons as:

$$H - \mu N = \frac{1}{2}\sum_{\mathbf{k}}(c^{\dagger}_{\mathbf{k}}c_{-\mathbf{k}})\mathcal{H}_{BdG}(\mathbf{k})\begin{pmatrix} c_{\mathbf{k}} \\ c^{\dagger}_{-\mathbf{k}} \end{pmatrix}, \tag{4}$$

where $\mathcal{H}_{BdG} = (\mathcal{H}_0(\mathbf{k}) - \mu)\sigma_z + \triangle_1(\mathbf{k})\sigma_x + \triangle_2(\mathbf{k})\sigma_y$, in terms of the Pauli matrices $\vec{\sigma}$, and $\triangle = \triangle_1 + i\triangle_2$ is the BCS mean field pairing potential, which for spin-free systems necessarily satisfies odd parity such that $\triangle(-\mathbf{k}) = -\triangle(\mathbf{k})$. The Bogolyubov de Gennes equation, $\mathcal{H}_{BdG}$, can be more expressed in terms of its intrinsic Particle-Hole symmetries as:

$$\Xi\mathcal{H}_{BdG}\Xi^{-1} = -\mathcal{H}_{BdG}(-\mathbf{k}), \tag{5}$$

such that $\Xi = \sigma_x K$ satisfying $\Xi^2 = +1$ and $\mathcal{H}_0(-\mathbf{k}) = \mathcal{H}_0(\mathbf{k})^*$. We see from this equation that each energy Eigenstate E of $\mathcal{H}_{BdG}$ has a coupling at $-E$: an obvious redundancy, since the Bogolyubov quasiparticle operators satisfy $\Gamma^{\dagger}_E = \Gamma_{-E}$, such that creating a quasiparticle at E is equivalent to annihilating one at $-E$.

Thus it can be shown that there are finite energy pairs for boundary states at $E = 0$ are protected since they cannot be displaced from $E = 0$ which either has or hasn't a zero mode determined by a $\mathbb{Z}_2$ class of topologies within the 1D superconductor bulk. We can represent the quasiparticle operators $\Gamma^{\dagger}_0 = \Gamma_0$ in terms of a particle which is equivalent to its own anti-particle —the distinctive characteristic of a Majorana Fermion.

4 Majorana Fermions

Majorna Fermions emerge at the interface of S-wave Superconductors and Topological Insulators, but we may also note that these quasiparticles emerge near the vortex core in proximity-induced superconduction regions along the surfaces of a Topological Insulator[7]. Here we consider an island of a typical Superconductor atop a larger mass of some Topological Insulating material.

Majorana states may exist solely in odd parity vorticities, from which we may find the zero-energy BdD solutions for 2D Superconductors induced on such a surface. We may model trial functions near the centre of the so-called Abrikosov vortex:

$$\psi_v = C_v \varphi \, \exp\left\{ -\int_0^r \mathrm{d}r' \left[\mathrm{A}(r') + \frac{\triangle(r')}{\hbar v_F} \right] - \frac{i\pi}{4} \right\}, \tag{6}$$

and analogously for the edges of the Superconducting Island:

$$\psi_e = C_e \varphi \, \exp\left\{ -\int_0^r \mathrm{d}r' \left[\mathrm{A}(r') - \frac{\triangle(r')}{\hbar v_F} \right] - \frac{i\pi}{4} \right\}, \tag{7}$$

and we thus find Landau levels localised at $r < R$, where R is the radius of the island, and $\triangle$ is the renormalised Superconduction gap.

5 Spin Liquids

The exotic behaviour of frustrated quantum magnets has brought about the discovery of discretised topological states of matter as fractional excitations. These pairs of 1/2-spinons are Majorana Fermions, having been measured via inelastic neutron scattering experiment[1]. In real materials this set of 1/2-spin moments atop a Honeycomb lattice can be modelled in terms of a Heisenberg interaction (J) with a Kitaev coupling K so that we may define the Hamiltonian as:

$$H = \sum_{i,j} \left(K S_i^m S_j^m + J \mathbf{S}_i \cdot \mathbf{S}_j \right), \tag{8}$$

such that the spin-component m is directed along the bond of the spins (i, j), which is stable for weak Heisenberg perturbations. By a pure Kitaev model we may fractionalise spin excitations into static fluxes and dynamical propagations of Majorana Fermions which are coupled minimally to a $\mathbb{Z}_2$ gauge field. This leads to an exactly solvable model of a 2D Quantum Spin Liquid State on the Honeycomb lattice. By the calibration of experimental apparata in the observation of QSL behaviour in particular hexagonal atomic material to the structural Bragg peaks, the ordered moments demonstrate strong spin fluctuation moments consistent with the model of a near liquid-like quantum state present in the material.

6 Conclusion

These emerging states of Topological matter are exciting for their theoretical depth and potential applications. It is well-known that it is possible to combine a Topological Insulator with common Superconductors to induce a correlated interface state such that this boundary obeys Majorana Fermion statistics.

Additionally, a well-separated pair of coupled Majorana bound states represents a qubit as a degenerate two level system[3]. By interchanging or braiding vortices adiabatically it is possible to generate quantum N qubit memory states by $2N$ Majorana bound states. Since we may consider the unitary operations on the state vector $|\psi_a\rangle \rightarrow U_{ab} |\psi_b\rangle$ which generalise the usual quantum statistics of Bosons and Fermions. Vis-à-vis three primary operations, (i) create, (ii) braid, and (iii) measure, we are able to perform all of the expected operations of a quantum computer. While as yet this suite of operations may not properly emulate a universal quantum computer, it is possible to protect quantum information topologically by the observation of non-Abelian statistics.

It is possible to predict electronic Topological Superconductors by their band structures theoretically and classify them according to their invariant topological properties. Due to the many correlated states of Topological Insulators and Superconductors there remains yet many new discoveries to be made. The emergence of experimental evidence of Majorana Fermions serves as proof of the value of a theoretical approach to practical physics problems.

As such, the field of Topological Insulators and Superconductors is rapidly rising to prominence as a booming subfield of condensed matter physics to be driven by both theoretical insight and experimental trial pursuit.

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